Errata for "A Cavendish Quantum Mechanics Primer"

Various small errors in Chapter 1 have been corrected in the on-line version, including:

- Page 5: "a generation later discovered *the* electron".
- Exercise 1.22: The correct answer for the sum of the two waves is $2\sin(kx)\cos(\phi/2)$.

Index: A mysterious latex error meant that the page for an index entry is wrong by either +1 or +2 pages. A corrected index is available from the errata web page and within the on-line version of Chapter 1.

Particles incident on a step, Page 79:
There are errors in the reflection coefficient A_- (a sign) and in the transmission coefficient A. Subsequent errors also arise.

- The A coefficients should be:

$$\begin{array}{lcll} \text{Reflection} & A_- & = & -A_+\dfrac{k' + \mathrm{i}k}{k' - \mathrm{i}k} \\ \text{Transmission} & A & = & -A_+\dfrac{2ik}{k' - \mathrm{i}k} \end{array}$$

- In the paragraph above Ex. 4.20, one has θ being the argument (note − sign) of
$$-(k' + \mathrm{i}k)/(k' - \mathrm{i}k).$$

- In Ex. 4.20 the phase to derive is rather:
$\theta = \pi + \arctan\left(2kk'/(k'^2 - k^2)\right) = \pi + \arctan\left(2\sqrt{E(V_0 - E)}/(V_0 - 2E)\right).$

  The discussion of the phase shift has been modified in the question and in the hints.

- In Ex.4.21 and in the paragraph above it, the prefactor being discussed is
$$-\frac{2ik}{k' - \mathrm{i}k}.$$

Corrected forms of pages 79 and 80 are attached to this erratum and available as part of the initial material, chapter 1, index pdf.

This is as expected from de Broglie, and also from Sturm–Liouville: above the step, lower kinetic energy means less curvature, nodes further spaced, and thus longer wavelength.

The conditions of continuity of ψ and of $\mathrm{d}\psi/\mathrm{d}x$ (wavefunction matching) allow us to solve the problem entirely. Consult the solution of the finite square well for the method and for the conditions on ψ — see around Eqs. (3.5–3.7) on page 51. Considering cases A and B in parallel, we have for the continuity:

$$A_+ + A_- = A \qquad B_+ + B_- = B$$
$$\mathrm{i}k(A_+ - A_-) = -k'A \qquad \mathrm{i}k(B_+ - B_-) = \mathrm{i}k''B$$

which can be solved for the weights of the various component wavefunctions:

$$\text{Reflection} \qquad A_- = -A_+\frac{k' + \mathrm{i}k}{k' - \mathrm{i}k} \qquad B_- = B_+\frac{k - k''}{k + k''}$$
$$\text{Transmission} \qquad A = -A_+\frac{2ik}{k' - \mathrm{i}k} \qquad B = B_+\frac{2k}{k + k''}.$$

We have solved for A_-, A, B_- and B, that is the scales of the reflected and transmitted waves, in terms of the incident wave amplitudes A_+ or B_+. Although the A and B results look superficially similar, they in fact differ qualitatively.

Firstly, in case A, the reflected wave amplitude modulus squared is $|A_-|^2 = |A_+|^2|\frac{k'+\mathrm{i}k}{k'-\mathrm{i}k}|^2$. The latter factor is $\frac{k'+\mathrm{i}k}{k'-\mathrm{i}k}\left(\frac{k'+\mathrm{i}k}{k'-\mathrm{i}k}\right)^*$ which becomes $\frac{k'+\mathrm{i}k}{k'-\mathrm{i}k}\frac{k'-\mathrm{i}k}{k'+\mathrm{i}k}$ on reversing the signs of i to get the complex conjugate (the *); overall the resultant number is clearly $= 1$. Therefore A_- and A_+ only differ in argument, but not modulus. Thus $A_- = A_+e^{\mathrm{i}\theta}$, where θ is the argument of $-(k' + \mathrm{i}k)/(k' - \mathrm{i}k)$. The reflected wave is the same magnitude as the incident wave (total reflection), but is shifted from it in phase. Note that we have total *external* reflection (in contrast to total internal reflection as in optics).

Exercise 4.20: Show that the phase shift (argument) of the totally reflected wave with respect to the incident wave is given by $\theta = \pi + \arctan\left(2kk'/(k'^2 - k^2)\right) = \pi + \arctan\left(2\sqrt{E(V_0 - E)}/(V_0 - 2E)\right)$ on substituting for k and k'. Examine θ as a function of E/V_0, the incident energy normalised by the step energy, around $E/V_0 \approx 0$, $\approx \frac{1}{2}$ and ≈ 1.

Hint: Expressions such as $-(k' + \mathrm{i}k)/(k' - \mathrm{i}k)$ need to have their complex factors taken into the numerator. Thus multiply top and bottom by the complex conjugate of the denominator. The real and imaginary parts are now easy to identify and share a common denominator which cancels on taking Im/Re; see Eq. (4.8). Recall that -1 can be thought of as having unit modulus and argument π; see Exs. 4.1 and 4.6, and Fig. 4.1. For $E \ll V_0$ the phase shift is $\sim \pi$, an inversion as for classical waves returning after complete reflection.

The wavefunction that penetrates the classically forbidden region under the step differs in both modulus and phase from the incident wave since the pre-factor $-2\mathrm{i}k/(k' - \mathrm{i}k)$ to the exponential is not of unit modulus and is not real. Since $\psi_A \sim \mathrm{e}^{-k'x}$, it extends a characteristic distance $d' \sim 1/k' = \hbar/\sqrt{2m(V_0 - E)}$. Thus as the incident energy E approaches the barrier height V_0, the wave penetrates ever further into the step.

Exercise 4.21: Find the modulus and phase of $-2\mathrm{i}k/(k' - \mathrm{i}k)$, and hence the connection between the evanescent (exponentially decaying) wave and the incident wave. Confirm that the wave in the forbidden region carries no current of particles into the potential step. The reflection is therefore total.

Secondly, in case B, the transmitted wave is now complex and can carry current. The reflected wave no longer differs in phase from the incident wave since $B_-/B_+ = (k - k'')/(k + k'')$ is purely real. This ratio is clearly less than 1 and the diminished reflected amplitude compared with the incident wave is depicted in Fig. 4.4B. The transmitted wave also has no phase shift since the ratio $B/B_+ = 2k/(k + k'')$ is real. The figure also emphasises the longer wavelength from the diminished, but still positive, kinetic energy.

By putting the various waves into the expression (4.21) for the current j, one can show that $j_+ = |B_+|^2 \frac{\hbar k}{m}$, $j_- = -|B_-|^2 \frac{\hbar k}{m}$ and $j_B = |B|^2 \frac{\hbar k''}{m}$. Note the sign in j_-; quantum mechanical current flows backwards (in negative x direction) in this wave (see Ex.4.19). Note each j carries a measure of its intensity (modulus squared) and the relevant wave vector factor (k or k''), appearing in the velocity-like combination $\hbar k/m$ that we discussed after Eq. (4.21).

Exercise 4.22: Check that the currents conserve overall flow of particles, that is $j_+ + j_- = j_B$. In effect, the *net* flow of current to the right before and after